10
Binomial Theorem

HISTORY The expression of the form $(a+b)^n$ is called a binomial. By direct multiplication it is easy to expand $(a+b)^2, (a+b)^3$, etc. The ancient mathematicians knew about the expansion of $(a+b)^n$ for $0 \le n \le 7$. Around 1660, B Pascal introduced Pascal's triangle for the coefficients in the expansion of $(a+b)^n$ and in the same year he gave the present form of binomial theorem.

BINOMIAL THEOREM (For Non-negative Integers)

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_{n-1} ab^{n-1} + {}^nC_n b^n.$$

By direct multiplication, we have

(i) $(a+b)^0 = 1.$ [$\because$ $x^0 = 1$ for every real x]

(ii) $(a+b)^1 = a+b = {}^1C_0 a + {}^1C_1 b.$

(iii) $(a+b)^2 = a^2 + 2ab + b^2 = {}^2C_0 a^2 + {}^2C_1 ab + {}^2C_2 b^2.$

(iv) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = {}^3C_0 a^3 + {}^3C_1 a^2b + {}^3C_2 ab^2 + {}^3C_3 b^3.$

(v) $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$= 4C_0 a^4 + {}^4C_1 a^3b + {}^4C_2 a^2b^2 + {}^4C_3 ab^3 + {}^4C_4 b^4.$

Given below is the ***Pascal's Triangle*** showing the coefficients of various terms in a binomial expansion.

Index	Coefficients
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1

PASCAL'S TRIANGLE

Look at the pattern given below:

For $n = 4$, the various coefficients are ${}^4C_0, {}^4C_1, {}^4C_2, {}^4C_3, {}^4C_4$.

For $n = 5$, the various coefficients are ${}^5C_0, {}^5C_1, {}^5C_2, {}^5C_3, {}^5C_4, {}^5C_5$.

For $n = 6$, the various coefficients are ${}^6C_0, {}^6C_1, {}^6C_2, {}^6C_3, {}^6C_4, {}^6C_5, {}^6C_6$.

For $n = 7$, the various coefficients are ${}^7C_0, {}^7C_1, {}^7C_2, {}^7C_3, {}^7C_4, {}^7C_5, {}^7C_6, {}^7C_7$.

BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

THEOREM *If a and b are real numbers then for all $n \in N$*

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_{n-1} ab^{n-1} + {}^nC_n b^n.$$

PROOF We shall prove the theorem by using the principle of mathematical induction.

Let $P(n)$ be the statement:

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_{n-1} ab^{n-1} + {}^nC_n b^n.$$

Then, the statement $P(1)$ is

$(a+b)^1 = {}^1C_0 a^1 + {}^1C_1 a^0 b = (a+b)$, which is true.

Thus, $P(1)$ is true.

Let $P(m)$ be true. Then,

$$(a+b)^m = {}^mC_0 a^m + {}^mC_1 a^{m-1}b + {}^mC_2 a^{m-2}b^2 + \ldots + {}^mC_{m-1} ab^{m-1} + {}^mC_m b^m. \quad \ldots \text{(i)}$$

Multiplying both sides of (i) by $(a+b)$, we get

$$\begin{aligned}(a+b)^{m+1} &= {}^mC_0 a^{m+1} + {}^mC_0 a^m b + {}^mC_1 a^m b + {}^mC_1 a^{m-1}b^2 + {}^mC_2 a^{m-1}b^2 \\ &\quad + {}^mC_2 a^{m-2}b^3 + \ldots + {}^mC_{m-1} a^2 b^{m-1} + {}^mC_{m-1} ab^m + {}^mC_m ab^m \\ &\quad + {}^mC_m b^{m+1} \\ &= {}^mC_0 a^{m+1} + ({}^mC_0 + {}^mC_1) a^m b + ({}^mC_1 + {}^mC_2) a^{m-1}b^2 + \ldots \\ &\quad + ({}^mC_{m-1} + {}^mC_m) ab^m + {}^mC_m b^{m+1} \\ &= {}^{m+1}C_0 a^{m+1} + {}^{m+1}C_1 a^m b + {}^{m+1}C_2 a^{m-1}b^2 + \ldots + {}^{m+1}C_m ab^m \\ &\quad + {}^{m+1}C_{m+1} b^{m+1}\end{aligned}$$

$[\because {}^mC_0 = 1 = {}^{m+1}C_0, {}^mC_m = 1 = {}^{m+1}C_{m+1}$ and ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r]$

This shows that $P(m+1)$ is true, whenever $P(m)$ is true.

Hence, by the principle of mathematical induction the theorem is true for all $n \in N$.

REMARK ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are known as binomial coefficients.

We may write the above result as:

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r.$$

SOME OBSERVATIONS IN A BINOMIAL EXPANSION

(i) The expansion of $(a+b)^n$ has $(n+1)$ terms.

(ii) Since ${}^nC_r = {}^nC_{n-r}$, we have ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$, and so on.

Thus, the coefficients of the terms equidistant from the beginning and the end in a binomial expansion are equal.

(iii) General term in the expansion of $(a+b)^n$ is the $(r+1)$th term given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

(iv) In the expansion of $(a+b)^n$, there are $(n+1)$ terms.

I. When n is even, the middle term $= \left(\frac{n}{2}+1\right)$th term.

II. When n is odd, two middle terms are

$\frac{1}{2}(n+1)$th term and $\left\{\frac{1}{2}(n+1)+1\right\}$th term.

(v) pth term from the end in $(a+b)^n$

$= (n+1-p+1)$th term from the beginning

$= (n-p+2)$th term from the beginning.

(vi) **Results on binomial coefficients**

We have the binomial expansion:

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \ldots + {}^nC_{n-1} a b^{n-1} + {}^nC_n b^n \quad \ldots (A)$$

Putting $a = b = 1$ in (A), we get

$${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n. \quad \ldots (B)$$

Putting $a = 1$ and $b = -1$ in (A) and using (B), we get

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = \frac{2^n}{2} = 2^{n-1}.$$

Hence, ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$.

DEDUCTIONS

(i) Putting $a = 1$ and $b = x$ in the binomial expansion of $(a+b)^n$, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n.$$

Thus, $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$.

And, $T_{r+1} = {}^nC_r x^r$.

(ii) Replacing b by $-b$ in the binomial expansion of $(a+b)^n$, we get

$$(a-b)^n = {}^nC_0 a^n - {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 - {}^nC_3 a^{n-3} b^3 + \ldots$$

$$\Rightarrow \quad (a-b)^n = \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r a^{n-r} b^r.$$

$\therefore$ T_{r+1} in the expansion of $(a-b)^n$ is given by

$$T_{r+1} = (-1)^r \cdot {}^nC_r a^{n-r} b^r.$$

(iii) Putting $a = 1$ and $b = x$ in the above expansion, we get

$$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots + (-1)^n \cdot {}^nC_n x^n$$

$$\Rightarrow \quad (1-x)^n = \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r x^r.$$

$\therefore$ T_{r+1} in the expansion of $(1-x)^n$ is given by

$$T_{r+1} = (-1)^r \cdot {}^nC_r x^r.$$

SUMMARY

1. We have

   (i) $(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r}b^r$ and $T_{r+1} = {}^nC_r a^{n-r}b^r$.

   (ii) $(a-b)^n = \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r a^{n-r}b^r$ and $T_{r+1} = (-1)^r \cdot {}^nC_r a^{n-r}b^r$.

   (iii) When n is even, then
   the middle term $= \left(\frac{n}{2}+1\right)$th term.

   (iv) When n is odd, then
   the middle terms are $\frac{1}{2}(n+1)$th and $\left\{\frac{1}{2}(n+1)+1\right\}$th.

   (v) pth term from the end $= (n-p+2)$th term from the beginning.

2. (i) $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$ and $T_{r+1} = {}^nC_r x^r$.

   (ii) $(1-x)^n = \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r x^r$ and $T_{r+1} = (-1)^r \cdot {}^nC_r x^r$.

3. (i) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$.

   (ii) ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$.

SOLVED EXAMPLES

EXAMPLE 1 *Expand* $(x^2+2y)^5$ *using binomial expansion.*

SOLUTION Using the binomial expansion, we have

$$(x^2+2y)^5 = {}^5C_0(x^2)^5 + {}^5C_1(x^2)^4(2y) + {}^5C_2(x^2)^3(2y)^2 + {}^5C_3(x^2)^2(2y)^3 + {}^5C_4x^2(2y)^4 + {}^5C_5(2y)^5$$
$$= x^{10} + 10x^8y + 40x^6y^2 + 80x^4y^3 + 80x^2y^4 + 32y^5.$$

EXAMPLE 2 *Expand* $\left(x^3 - \frac{2}{x^2}\right)^6$ *using binomial expansion.*

SOLUTION Using the binomial expansion, we get

$$\left(x^3 - \frac{2}{x^2}\right)^6 = {}^6C_0(x^3)^6 - {}^6C_1(x^3)^5\left(\frac{2}{x^2}\right) + {}^6C_2(x^3)^4\left(\frac{2}{x^2}\right)^2 - {}^6C_3(x^3)^3\left(\frac{2}{x^2}\right)^3 + {}^6C_4(x^3)^2\left(\frac{2}{x^2}\right)^4 - {}^6C_5x^3\left(\frac{2}{x^2}\right)^5 + {}^6C_6\left(\frac{2}{x^2}\right)^6$$
$$= x^{18} - 12x^{13} + 60x^8 - 160x^3 + \frac{240}{x^2} - \frac{192}{x^7} + \frac{64}{x^{12}}.$$

EXAMPLE 3 *Expand* $(1+x+x^2)^3$ *using binomial expansion.*

SOLUTION Put $1+x = y$. Then,

$$(1+x+x^2)^3 = (y+x^2)^3$$

$$= {}^3C_0 y^3 + {}^3C_1 y^2 x^2 + {}^3C_2 y (x^2)^2 + {}^3C_3 (x^2)^3$$
$$= y^3 + 3y^2x^2 + 3yx^4 + x^6$$
$$= (1+x)^3 + 3x^2(1+x)^2 + 3x^4(1+x) + x^6 \qquad [\because\ y = (1+x)]$$
$$= (1 + {}^3C_1 x + {}^3C_2 x^2 + {}^3C_3 x^3) + 3x^2(1 + 2x + x^2) + 3x^4(1+x) + x^6$$
$$= (1 + 3x + 3x^2 + x^3) + (3x^2 + 6x^3 + 3x^4) + (3x^4 + 3x^5) + x^6$$
$$= x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1.$$

Hence, $(1 + x + x^2)^3 = x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1.$

EXAMPLE 4 *Expand* $(1 - x + x^2)^4$ *using binomial expansion.*

SOLUTION Put $1 - x = y$. Then,

$$(1 - x + x^2)^4 = (y + x^2)^4$$
$$= {}^4C_0 y^4 + {}^4C_1 y^3 x^2 + {}^4C_2 y^2 (x^2)^2 + {}^4C_3 y (x^2)^3 + {}^4C_4 (x^2)^4$$
$$= y^4 + 4y^3x^2 + 6y^2x^4 + 4yx^6 + x^8$$
$$= (1-x)^4 + 4(1-x)^3x^2 + 6(1-x)^2x^4 + 4(1-x)x^6 + x^8$$
$$[\because\ y = 1 - x]$$
$$= (1 - {}^4C_1 x + {}^4C_2 x^2 - {}^4C_3 x^3 + {}^4C_4 x^4)$$
$$+4x^2(1 - {}^3C_1 x + {}^3C_2 x^2 - {}^3C_3 x^3)$$
$$+6x^4(1 - 2x + x^2) + 4x^6(1 - x) + x^8$$
$$= (1 - 4x + 6x^2 - 4x^3 + x^4) + 4x^2(1 - 3x + 3x^2 - x^3)$$
$$+6x^4(1 - 2x + x^2) + 4x^6(1 - x) + x^8$$
$$= (1 - 4x + 6x^2 - 4x^3 + x^4) + (4x^2 - 12x^3 + 12x^4 - 4x^5)$$
$$+(6x^4 - 12x^5 + 6x^6) + (4x^6 - 4x^7) + x^8$$
$$= x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1.$$

Hence,

$$(1 - x + x^2)^4 = x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1.$$

EXAMPLE 5 *Expand* $\{(a+b)^4 + (a-b)^4\}$ *and use it to evaluate*

$$(x^2 + \sqrt{1-x^2})^4 + (x^2 - \sqrt{1-x^2})^4.$$

SOLUTION We have

$$(a+b)^4 + (a-b)^4 = [{}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 ab^3 + {}^4C_4 b^4]$$
$$+[{}^4C_0 a^4 - {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 - {}^4C_3 ab^3 + {}^4C_4 b^4]$$
$$= 2[{}^4C_0 a^4 + {}^4C_2 a^2 b^2 + {}^4C_4 b^4]. \qquad \ldots (i)$$

Putting $\sqrt{1-x^2} = y$, we get

$$(x^2 + \sqrt{1-x^2})^4 + (x^2 - \sqrt{1-x^2})^4$$
$$= (x^2 + y)^4 + (x^2 - y)^4$$
$$= 2[{}^4C_0 (x^2)^4 + {}^4C_2 (x^2)^2 y^2 + {}^4C_4 y^4]$$
$$[\text{taking } a = x^2 \text{ and } b = y \text{ in (i)}]$$
$$= 2[x^8 + 6x^4 y^2 + y^4]$$
$$= 2[x^8 + 6x^4(1-x^2) + (1-x^2)^2] \qquad [\because\ y^2 = (1-x^2)]$$

$$= 2[x^8 + (6x^4 - 6x^6) + (1 - 2x^2 + x^4)]$$
$$= 2[x^8 - 6x^6 + 7x^4 - 2x^2 + 1]$$
$$= 2x^8 - 12x^6 + 14x^4 - 4x^2 + 2.$$

EXAMPLE 6 *Find the 10th term in the expansion of* $\left(2x^2 + \frac{1}{x}\right)^{12}$.

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = {}^{12}C_r \times (2x^2)^{(12-r)} \times \left(\frac{1}{x}\right)^r. \qquad \ldots \text{(i)}$$

$\therefore \quad T_{10} = T_{(9+1)}$

$$= {}^{12}C_9 \times (2x^2)^{(12-9)} \times \left(\frac{1}{x}\right)^9 \qquad [\text{putting } r = 9 \text{ in (i)}]$$

$$= {}^{12}C_3 \times (2x^2)^3 \times \left(\frac{1}{x^9}\right) \qquad [\because {}^{12}C_9 = {}^{12}C_{(12-9)} = {}^{12}C_3]$$

$$= \left(\frac{12 \times 11 \times 10}{3 \times 2 \times 1} \times 8x^6 \times \frac{1}{x^9}\right) = \frac{1760}{x^3}.$$

Hence, the 10th term in the given expansion is $\frac{1760}{x^3}$.

EXAMPLE 7 *Find the 6th term in the expansion of* $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$.

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^9C_r \times \left(\frac{4x}{5}\right)^{(9-r)} \times \left(\frac{5}{2x}\right)^r. \qquad \ldots \text{(i)}$$

$\therefore \quad T_6 = T_{(5+1)}$

$$= (-1)^5 \times {}^9C_5 \times \left(\frac{4x}{5}\right)^{(9-5)} \times \left(\frac{5}{2x}\right)^5 \qquad [\text{putting } r = 5 \text{ in (i)}]$$

$$= -{}^9C_4 \times \left(\frac{4x}{5}\right)^4 \times \left(\frac{5}{2x}\right)^5 \qquad [\because {}^9C_5 = {}^9C_{(9-5)} = {}^9C_4]$$

$$= -\left[\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times \frac{2^8 x^4}{5^4} \times \frac{5^5}{2^5 x^5}\right] = \frac{-5040}{x}.$$

Hence, the 6th term in the given expansion is $\frac{-5040}{x}$.

EXAMPLE 8 *Find the 4th term from the end in the expansion of* $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^9$.

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^9C_r \times \left(\frac{x^3}{2}\right)^{(9-r)} \times \left(\frac{2}{x^2}\right)^r. \qquad \ldots \text{(i)}$$

Now, pth term from the end

$= (n - p + 2)$th term from the beginning.

$\therefore$ 4th term from the end

$= (9 - 4 + 2)$th term from the beginning $\quad [\because n = 9, p = 4]$

$=$ 7th term from the beginning

$$= T_7 = T_{(6+1)}$$

$$= (-1)^6 \times {}^9C_6 \times \left(\frac{x^3}{2}\right)^{(9-6)} \times \left(\frac{2}{x^2}\right)^6 \qquad \text{[putting } r = 6 \text{ in (i)]}$$

$$= {}^9C_3 \times \left(\frac{x^3}{2}\right)^3 \times \left(\frac{2}{x^2}\right)^6 \qquad [\because\ {}^9C_6 = {}^9C_{(9-6)} = {}^9C_3]$$

$$= \left(\frac{9\times 8\times 7}{3\times 2\times 1} \times \frac{x^9}{8} \times \frac{64}{x^{12}}\right) = \frac{672}{x^3}.$$

Hence, the 4th term from the end is $\frac{672}{x^3}$.

EXAMPLE 9 *If the ratio of the 5th term from the beginning to the 5th term from the end in the expansion of* $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ *is* $\sqrt{6} : 1$ *then find the value of n.*

SOLUTION The given expression may be written as $(2^{\frac{1}{4}} + 3^{-\frac{1}{4}})^n$.

The general term in this expansion is given by

$$T_{r+1} = {}^nC_r \times (2^{\frac{1}{4}})^{(n-r)} \times (3^{-\frac{1}{4}})^r$$

$$\Rightarrow \quad T_{r+1} = {}^nC_r \times 2^{\frac{(n-r)}{4}} \times 3^{\left(\frac{-r}{4}\right)}. \qquad \ldots \text{(i)}$$

Now, pth term from the end

$= (n - p + 2)$th term from the beginning.

$\therefore$ 5th term from the end

$= (n - 5 + 2)$th term from the beginning

$= (n - 3)$th term from the beginning.

Now, $T_5 = T_{(4+1)}$

$$= {}^nC_4 \times 2^{\frac{(n-4)}{4}} \times 3^{\left(\frac{-4}{4}\right)} \qquad \text{[putting } r = 4 \text{ in (i)]}$$

$$= {}^nC_4 \times 2^{\frac{(n-4)}{4}} \times 3^{-1}. \qquad \ldots \text{(ii)}$$

And, $T_{n-3} = T_{(n-4)+1}$

$$= {}^nC_{n-4} \times 2^1 \times 3^{\frac{-(n-4)}{4}} \qquad \text{[putting } r = (n-4) \text{ in (i)]}$$

$$= {}^nC_4 \times 2 \times 3^{\frac{-(n-4)}{4}}. \qquad \ldots \text{(iii)} \quad [\because\ {}^nC_{n-r} = {}^nC_r]$$

$$\therefore \quad \frac{T_5}{T_{n-3}} = \frac{\sqrt{6}}{1}$$

$$\Rightarrow \quad \frac{{}^nC_4 \times 2^{\frac{(n-4)}{4}} \times 3^{-1}}{{}^nC_4 \times 2 \times 3^{\frac{-(n-4)}{4}}} = \frac{\sqrt{6}}{1}$$

$$\Rightarrow \quad 2^{\left\{\frac{(n-4)}{4}-1\right\}} \times 3^{\left\{\frac{(n-4)}{4}-1\right\}} = \sqrt{6}$$

$$\Rightarrow \quad 2^{\frac{(n-8)}{4}} \times 3^{\frac{(n-8)}{4}} = 6^{1/2}$$

$$\Rightarrow \quad (2\times 3)^{\frac{(n-8)}{4}} = 6^{1/2}$$

$$\Rightarrow \quad \frac{n-8}{4} = \frac{1}{2} \Rightarrow n-8 = 2 \Rightarrow n = 10.$$

Hence, $n = 10$.

EXAMPLE 10 *Find the term independent of x in the expansion of* $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^{15}$.

SOLUTION Let T_{r+1} be independent of x.

Here, $T_{r+1} = (-1)^r \times {}^{15}C_r \times \left(\frac{3x^2}{2}\right)^{(15-r)} \times \left(\frac{1}{3x}\right)^r$

$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^{15}C_r \times 3^{(15-2r)} \times 2^{(r-15)} \times x^{(30-3r)}. \qquad \ldots (i)$$

Now, T_{r+1} will be independent of x only when the power of x in it is 0.

$\therefore \quad 30-3r = 0 \Rightarrow 3r = 30 \Rightarrow r = 10 \Rightarrow r+1 = 11.$

Thus, T_{11} is independent of x.

Now, $T_{11} = T_{(10+1)}$

$= (-1)^{10} \times {}^{15}C_{10} \times 3^{(15-20)} \times 2^{(10-15)} \times x^0$ [putting $r = 10$ in (i)]

$= {}^{15}C_5 \times 3^{-5} \times 2^{-5} = \frac{{}^{15}C_5}{6^5} \qquad \left[\because {}^{15}C_{10} = {}^{15}C_{(15-10)} = {}^{15}C_5\right]$

$= \left(\frac{15\times 14\times 13\times 12\times 11}{5\times 4\times 3\times 2\times 1} \times \frac{1}{36\times 36\times 6}\right) = \frac{1001}{2592}.$

Hence, the term independent of x in the given expansion is $\frac{1001}{2592}$.

EXAMPLE 11 *Find the term independent of x in the expansion of* $\left(\frac{\sqrt{x}}{\sqrt{3}} + \frac{\sqrt{3}}{2x^2}\right)^{10}$.

SOLUTION Let T_{r+1} be independent of x.

Here, $T_{r+1} = {}^{10}C_r \times \left(\sqrt{\frac{x}{3}}\right)^{(10-r)} \times \left(\frac{\sqrt{3}}{2x^2}\right)^r$

$$\Rightarrow \quad T_{r+1} = {}^{10}C_r \times \left(\frac{x}{3}\right)^{\frac{(10-r)}{2}} \times 3^{\frac{r}{2}} \times \frac{1}{2^r x^{2r}}$$

$$= {}^{10}C_r \times x^{\left(5-\frac{r}{2}-2r\right)} \times \frac{1}{3^{\left(5-\frac{r}{2}\right)}} \times 3^{\frac{r}{2}} \times 2^{-r}$$

$$= {}^{10}C_r \times x^{\left(5-\frac{5r}{2}\right)} \times 3^{(r-5)} \times 2^{-r}. \qquad \ldots (i)$$

Now, T_{r+1} will be independent of x only when the power of x in it is 0.

$\therefore \quad 5-\frac{5r}{2}=0 \Rightarrow \frac{5r}{2}=5 \Rightarrow r=2 \Rightarrow r+1=3.$

Thus, T_3 is independent of x.

Putting $r=2$ in (i), we get

$$T_3 = T_{(2+1)}$$

$$= {}^{10}C_2 \times 3^{(2-5)} \times 2^{-2} \times x^0 = \left(\frac{10\times 9}{2\times 1} \times 3^{-3} \times 2^{-2}\right)$$

$$= \left(\frac{45}{3^3 \times 2^2}\right) = \left(\frac{45}{27\times 4}\right) = \frac{5}{12}.$$

Hence, the term independent of x in the given expansion is $\frac{5}{12}$.

EXAMPLE 12 *If the term free from x in the expansion of* $\left(\sqrt{x}-\frac{m}{x^2}\right)^{10}$ *is* 405, *find the value of m.*

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^{10}C_r \times (\sqrt{x})^{(10-r)} \times \left(\frac{m}{x^2}\right)^r$$

$$= (-1)^r \times {}^{10}C_r \times x^{\left(5-\frac{r}{2}\right)} \times \frac{m^r}{x^{2r}}$$

$$= (-1)^r \times {}^{10}C_r \times x^{\left(5-\frac{r}{2}-2r\right)} \times m^r$$

$$= (-1)^r \times {}^{10}C_r \times x^{\left(5-\frac{5r}{2}\right)} \times m^r. \qquad \ldots \text{(i)}$$

Let T_{r+1} be free from x.

Then, the power of x in T_{r+1} must be 0.

$\therefore \quad 5-\frac{5r}{2}=0 \Rightarrow \frac{5r}{2}=5 \Rightarrow r=2 \Rightarrow r+1=3.$

So, T_3 will be free from x.

Now, $T_3 = T_{(2+1)}$

$$= (-1)^2 \times {}^{10}C_2 \times x^0 \times m^2 \qquad \text{[putting } r=2 \text{ in (i)]}$$

$$= \left(\frac{10\times 9}{2} \times m^2\right) = 45m^2.$$

But, it is given that the term free from x is 405.

$\therefore \quad 45m^2 = 405 \Rightarrow m^2 = 9 \Rightarrow m = \pm 3.$

Hence, $m = \pm 3$.

EXAMPLE 13 *Find the coefficients of* x^{32} *and* $\frac{1}{x^{17}}$ *in the expansion of* $\left(x^4-\frac{1}{x^3}\right)^{15}$.

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^{15}C_r \times (x^4)^{(15-r)} \times \left(\frac{1}{x^3}\right)^r$$

$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^{15}C_r \times x^{(60-7r)}. \qquad \ldots \text{(i)}$$

Putting $60-7r=32$, we get $7r=28 \Rightarrow r=4 \Rightarrow r+1=5$.

Now, $T_5 = T_{(4+1)} = (-1)^4 \times {}^{15}C_4 \times x^{(60-28)} = {}^{15}C_4 \times x^{32}$.

$\therefore$ coefficient of $x^{32} = {}^{15}C_4 = \left(\frac{15\times14\times13\times12}{4\times3\times2\times1}\right) = 1365.$

Let T_{s+1} be the term containing x^{-17} $\left[\because \frac{1}{x^{17}} = x^{-17}\right]$.

Then, $T_{s+1} = (-1)^s \times {}^{15}C_s \times x^{(60-7s)}$. … (ii) [putting $r=s$ in (i)]

Putting $60-7s=-17$, we get $s=11$ and therefore, $s+1=12$.

Thus, the 12th term contains x^{-17}.

Now, $T_{12} = T_{(11+1)}$

$= (-1)^{11} \times {}^{15}C_{11} \times x^{(60-77)}$ [putting $s=11$ in (ii)]

$= -{}^{15}C_4 \times x^{-17}$.

$\therefore$ coefficient of $x^{-17} = -{}^{15}C_4 = -\left(\frac{15\times14\times13\times12}{4\times3\times2\times1}\right) = -1365.$

So, the coefficient of x^{-17} in the given expansion is –1365.

Hence, the coefficient of x^{32} is 1365 and that of x^{-17} is –1365.

EXAMPLE 14 *Prove that there is no term containing x^{10} in the expansion of $\left(x^2-\frac{2}{x}\right)^{18}$.*

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^{18}C_r \times (x^2)^{(18-r)} \times \left(\frac{2}{x}\right)^r$$

$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^{18}C_r \times 2^r \times x^{(36-3r)}.$$

Let T_{r+1} contain x^{10}. Then,

$$36-3r=10 \Rightarrow 3r=26 \Rightarrow r=\frac{26}{3}.$$

Since the value of r cannot be a fraction, so there is no term containing x^{10}.

EXAMPLE 15 *Find the middle term in the expansion of $\left(ax-\frac{b}{x^2}\right)^{12}$.*

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^{12}C_r \times (ax)^{(12-r)} \times \left(\frac{b}{x^2}\right)^r$$

$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^{12}C_r \times a^{(12-r)} \times b^r \times x^{(12-3r)}. \quad \text{… (i)}$$

Since the given binomial contains an even power, so it has only one middle term.

Middle term $= \left(\frac{12}{2}+1\right)$th term = 7th term.

Now, $T_7 = T_{(6+1)}$

$= (-1)^6 \times {}^{12}C_6 \times a^{(12-6)} \times b^6 \times x^{(12-3\times6)}$ [putting $r=6$ in (i)]

$= {}^{12}C_6 \times a^6 b^6 \times x^{-6}$

$$= \left(\frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \times \frac{a^6 b^6}{x^6} \right) = \frac{924 a^6 b^6}{x^6}.$$

Hence, the middle term in the given expansion is $\frac{924 a^6 b^6}{x^6}$.

EXAMPLE 16 *Find the middle terms in the expansion of* $\left(3x - \frac{x^3}{6}\right)^7$.

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^7C_r \times (3x)^{7-r} \times \left(\frac{x^3}{6}\right)^r$$

$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^7C_r \times 3^{(7-2r)} \times 2^{-r} \times x^{(7+2r)}. \qquad \ldots \text{(i)}$$

Clearly, the given expansion has 8 terms.

So, it has two middle terms, namely $\left(\frac{8}{2}\right)$th, i.e., 4th and 5th.

Now, $T_4 = T_{(3+1)}$

$$= (-1)^3 \times {}^7C_3 \times 3^{(7-6)} \times 2^{-3} \times x^{13} \qquad [\text{putting } r = 3 \text{ in (i)}]$$

$$= -\left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times 3^1 \times \frac{1}{2^3} \times x^{13} \right)$$

$$= -\left(\frac{35 \times 3 \times x^{13}}{8} \right) = \frac{-105 x^{13}}{8}.$$

And, $T_5 = T_{(4+1)}$

$$= (-1)^4 \times {}^7C_4 \times 3^{(7-8)} \times 2^{-4} \times x^{15} \qquad [\text{putting } r = 4 \text{ in (i)}]$$

$$= {}^7C_3 \times 3^{-1} \times 2^{-4} \times x^{15} \qquad [\because {}^7C_4 = {}^7C_{(7-4)} = {}^7C_3]$$

$$= \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{1}{3} \times \frac{1}{2^4} \times x^{15} \right) = \frac{35 x^{15}}{48}.$$

Hence, the required middle terms are $\frac{-105 x^{13}}{8}$ and $\frac{35 x^{15}}{48}$.

EXAMPLE 17 *Show that the middle term in the expansion of* $(1+x)^{2n}$ *is* $\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{n!} \cdot 2^n \cdot x^n$, *where* $n \in N$.

SOLUTION Clearly, the number of terms in the expansion of $(1+x)^{2n}$ is $(2n+1)$.

$\therefore$ middle term $= \left(\frac{2n}{2} + 1\right)$th term $= (n+1)$th term $= T_{n+1}$.

Now, $T_{n+1} = {}^{2n}C_n \cdot x^n = \frac{(2n)!}{(n!) \times (n!)} \cdot x^n$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \ldots (2n-2)(2n-1)(2n)}{(n!) \times (n!)} \cdot x^n$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (2n-3)(2n-1)] \times [2 \cdot 4 \cdot 6 \ldots (2n-2)(2n)]}{(n!)(n!)} \cdot x^n$$

$$= \frac{[1\cdot 3\cdot 5\ldots(2n-1)]\times 2^n\times[1\cdot 2\cdot 3\ldots(n-1)\cdot n]}{(n!)\times(n!)}\cdot x^n$$

$$= \frac{[1\cdot 3\cdot 5\ldots(2n-1)]\times 2^n\times x^n}{(n!)}.$$

Hence, the middle term in the given expansion is

$$\frac{1\cdot 3\cdot 5\ldots(2n-1)\times 2^n\times x^n}{(n!)}.$$

EXAMPLE 18 *If x^p occurs in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$, prove that its coefficient is*

$$\frac{(2n)!}{\left\{\left(\frac{4n-p}{3}\right)!\right\}\times\left\{\left(\frac{2n+p}{3}\right)!\right\}}.$$

SOLUTION The general term in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$ is given by

$$T_{r+1} = {}^{2n}C_r\times(x^2)^{(2n-r)}\times\left(\frac{1}{x}\right)^r$$

$$\Rightarrow \quad T_{r+1} = {}^{2n}C_r\times x^{(4n-3r)}. \qquad \ldots \text{(i)}$$

This term contains x^p only when $4n-3r=p$.

And, $4n-3r=p \;\Rightarrow\; r=\frac{(4n-p)}{3}$.

Putting $4n-3r=p$ in (i), we get

coefficient of $x^p = {}^{2n}C_r$, where $r=\frac{(4n-p)}{3}$

$$= \frac{(2n)!}{(r!)\times(2n-r)!} = \frac{(2n)!}{\left\{\left(\frac{4n-p}{3}\right)!\right\}\times\left\{\left[2n-\frac{(4n-p)}{3}\right]!\right\}}$$

$$= \frac{(2n)!}{\left\{\left(\frac{4n-p}{3}\right)!\right\}\times\left\{\left(\frac{2n+p}{3}\right)!\right\}}.$$

Hence, the coefficient of x^p in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$ is

$$\frac{(2n)!}{\left\{\left(\frac{4n-p}{3}\right)!\right\}\times\left\{\left(\frac{2n+p}{3}\right)!\right\}}.$$

EXAMPLE 19 *Show that the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is the sum of the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$.*

SOLUTION The expansion of $(1+x)^{2n}$ contains $(2n+1)$ terms and therefore, the $(n+1)$th term is the middle term.

$\therefore$ the middle term in the expansion of $(1+x)^{2n}$ is given by

$$t_{n+1} = {}^{2n}C_n x^n = \frac{(2n)!}{(n!)\cdot(n!)}x^n = \frac{(2n)!}{(n!)^2}x^n.$$

Thus, the coefficient of the middle term of $(1+x)^{2n}$ is $\frac{(2n)!}{(n!)^2}$.

Again, the expansion of $(1+x)^{2n-1}$ contains $2n$ terms.

So, the middle terms are the nth and the $(n+1)$th terms.

Now, $t_n = t_{(n-1)+1} = {}^{2n-1}C_{n-1}x^{n-1}$ and $t_{n+1} = {}^{2n-1}C_n x^n$.

$\therefore$ the sum of the coefficients of the middle terms of $(1+x)^{2n-1}$

$$= {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$$

$$= \frac{(2n-1)!}{(n-1)!\times(n!)} + \frac{(2n-1)!}{(n!)\times(n-1)!}$$

$$= \frac{2\times(2n-1)!\times n}{(n!)\times n\times(n-1)!} \quad \text{[multiplying num. and denom. by } n\text{]}$$

$$= \frac{(2n)\cdot(2n-1)!}{(n!)\cdot(n!)} = \frac{(2n)!}{(n!)^2}$$

= the coefficient of the middle term of $(1+x)^{2n}$.

EXAMPLE 20 *If a_1, a_2, a_3, a_4 be the coefficients of four consecutive terms in the expansion of $(1+x)^n$ then prove that*

$$\frac{a_1}{(a_1+a_2)} + \frac{a_3}{(a_3+a_4)} = \frac{2a_2}{(a_2+a_3)}.$$

SOLUTION Let a_1, a_2, a_3 and a_4 be the coefficients of the rth, $(r+1)$th, $(r+2)$th and $(r+3)$th terms in the expansion of $(1+x)^n$. Then,

$$a_1 = {}^nC_{r-1};\ a_2 = {}^nC_r,\ a_3 = {}^nC_{r+1} \text{ and } a_4 = {}^nC_{r+2}.$$

Now, $a_1 + a_2 = ({}^nC_{r-1} + {}^nC_r) = {}^{n+1}C_r$;

$$a_3 + a_4 = ({}^nC_{r+1} + {}^nC_{r+2}) = {}^{n+1}C_{r+2};$$

and $a_2 + a_3 = ({}^nC_r + {}^nC_{r+1}) = {}^{n+1}C_{r+1}$.

$\therefore$

$$\frac{a_1}{(a_1+a_2)} + \frac{a_3}{(a_3+a_4)} = \frac{{}^nC_{r-1}}{{}^{n+1}C_r} + \frac{{}^nC_{r+1}}{{}^{n+1}C_{r+2}}$$

$$= \frac{{}^nC_{r-1}}{\left(\frac{n+1}{r}\right)\cdot{}^nC_{r-1}} + \frac{{}^nC_{r+1}}{\left(\frac{n+1}{r+2}\right)\cdot{}^nC_{r+1}} \quad \left[\because {}^nC_r = \left(\frac{n}{r}\right)\cdot{}^{n-1}C_{r-1}\right]$$

$$= \frac{r}{(n+1)} + \frac{(r+2)}{(n+1)} = \frac{2(r+1)}{(n+1)}.$$

And, $\frac{2a_2}{(a_2+a_3)} = \frac{2\times{}^nC_r}{{}^{n+1}C_{r+1}} = \frac{2\times{}^nC_r}{\left(\frac{n+1}{r+1}\right)\cdot{}^nC_r} = \frac{2(r+1)}{(n+1)}.$

Hence, $\frac{a_1}{(a_1+a_2)} + \frac{a_3}{(a_3+a_4)} = \frac{2a_2}{(a_2+a_3)}.$

EXAMPLE 21 *In the binomial expansion of* $(1+x)^{m+n}$, *prove that the coefficients of* x^m *and* x^n *are equal.*

SOLUTION In the binomial expansion of $(1+x)^{m+n}$, the general term is given by

$$T_{r+1} = {}^{m+n}C_r \times x^r. \qquad \ldots \text{(i)}$$

$\therefore$ coefficient of x^m in the expansion of $(1+x)^{m+n}$

$$= {}^{m+n}C_m = \frac{(m+n)!}{\{(m+n-m)!\} \times (m!)} = \frac{(m+n)!}{(n!) \times (m!)}.$$

coefficient of x^n in the expansion of $(1+x)^{m+n}$

$$= {}^{m+n}C_n = \frac{(m+n)!}{\{(m+n-n)!\} \times (n!)} = \frac{(m+n)!}{(m!) \times (n!)}.$$

Hence, the coefficients of x^m and x^n are equal.

EXAMPLE 22 *Find the value of r when it is being given that the coefficients of* $(2r+4)$*th and* $(r-2)$*th terms in the expansion of* $(1+x)^{18}$ *are equal.*

SOLUTION In the expansion of $(1+x)^{18}$, we have $T_{p+1} = {}^{18}C_p x^p$.

$\therefore$ $T_{(2r+4)} = T_{(2r+3)+1} = {}^{18}C_{2r+3} \cdot x^{2r+3}$ ... (i)

and $T_{(r-2)} = T_{(r-3)+1} = {}^{18}C_{(r-3)} \cdot x^{r-3}.$ ... (ii)

It is being given that the coefficients of $T_{(2r+1)}$ and $T_{(r-2)}$ are equal.

So, from (i) and (ii), we get

$${}^{18}C_{2r+3} = {}^{18}C_{r-3} \Rightarrow (2r+3) = (r-3) \text{ or } (2r+3)+(r-3) = 18$$

$[\because {}^nC_p = {}^nC_q \Rightarrow p = q \text{ or } p+q = n].$

Now, $(2r+3) = (r-3)$ or $(2r+3)+(r-3) = 18$

$\Rightarrow$ $r = -6$ or $r = 6$

$\Rightarrow$ $r = 6$ [$\because$ the negative value of r is not permissible].

Hence, $r = 6$.

EXAMPLE 23 *Find the coefficient of* x^6y^3 *in the expansion of* $(x+2y)^9$.

SOLUTION The general term in the expansion of $(x+2y)^9$ is given by

$$T_{r+1} = {}^9C_r \times x^{(9-r)} \times (2y)^r$$

$$\Rightarrow T_{r+1} = {}^9C_r \times 2^r \times x^{(9-r)} \times y^r. \qquad \ldots \text{(i)}$$

We have to find the coefficient of x^6y^3.

Putting $r = 3$ in (i), we get

$$T_{(3+1)} = {}^9C_3 \times 2^3 \times (x^6y^3).$$

$\therefore$ coefficient of x^6y^3 in the given expansion

$$= ({}^9C_3 \times 2^3) = \left(\frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times 8\right) = 672.$$

Hence, the coefficient of x^6y^3 in the given expansion is 672.

EXAMPLE 24 *Write down the binomial expansion of* $(1+x)^{n+1}$ *when* $x = 8$. *Deduce that* $(9^{n+1} - 8n - 9)$ *is divisible by* 64, *where n is a positive integer.*

SOLUTION Using the binomial expansion, we get

$$(1+x)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + {}^{n+1}C_3 x^3 + \ldots + {}^{n+1}C_{n+1} x^{n+1}.$$

Putting $x = 8$ in the above expansion, we get

$$9^{n+1} = 1 + (n+1) \times 8 + {}^{n+1}C_2 \times (8)^2 + {}^{n+1}C_3 \times (8)^3 + \ldots + {}^{n+1}C_{n+1} \times (8)^{n+1}$$

$$\Rightarrow \quad (9^{n+1} - 8n - 9) = (8)^2 \times \{{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \ldots + (8)^{n-1}\}$$

$$\Rightarrow \quad (9^{n+1} - 8n - 9) = 64 \times (\text{an integer}).$$

Hence, $(9^{n+1} - 8n - 9)$ is exactly divisible by 64.

EXAMPLE 25 *Using the binomial theorem, prove that* $(6^n - 5n)$ *always leaves the remainder* 1 *when divided by* 25.

SOLUTION Using binomial expansion, we get

$$\begin{aligned}(6^n - 5n) &= (1+5)^n - 5n \\ &= \{{}^nC_0 + {}^nC_1 \times 5 + {}^nC_2 \times (5)^2 + {}^nC_3 \times (5)^3 + {}^nC_4 \times (5)^4 + \ldots + {}^nC_n \times 5^n\} - 5n \\ &= \{1 + 5n + {}^nC_2 \times (5)^2 + {}^nC_3 \times (5)^3 + {}^nC_4 \times (5)^4 + \ldots + {}^nC_n \times 5^n\} - 5n \\ &= \{1 + {}^nC_2 \times (5)^2 + {}^nC_3 \times (5)^3 + {}^nC_4 \times (5)^4 + \ldots + {}^nC_n \times 5^n\} \\ &= (5)^2 \times \{{}^nC_2 + {}^nC_3 \times 5 + {}^nC_4 \times (5)^2 + \ldots + {}^nC_n \times 5^{(n-2)}\} + 1 \\ &= 25 \times (\text{an integer}) + 1.\end{aligned}$$

Hence, $(6^n - 5n)$ when divided by 25 leaves the remainder 1.

EXAMPLE 26 *Show that* $2^{4n+4} - 15n - 16$, *where* $n \in N$ *is divisible by* 225.

SOLUTION We have

$$2^{4n+4} - 15n - 16 = 2^{4(n+1)} - 15n - 16 = 16^{(n+1)} - 15n - 16.$$

Now, $16^{(n+1)} - 15n - 16 = (1+15)^{n+1} - 15n - 16$

$$\begin{aligned}&= {}^{n+1}C_0 + {}^{n+1}C_1 \times 15 + {}^{n+1}C_2 \times (15)^2 + {}^{n+1}C_3 \times (15)^3 + \ldots + {}^{n+1}C_{n+1} \times (15)^{n+1} - 15n - 16 \\ &= 1 + (n+1) \times 15 + {}^{n+1}C_2 \times (15)^2 + {}^{n+1}C_3 \times (15)^3 + \ldots + (15)^{n+1} - 15n - 16 \\ &= {}^{n+1}C_2 \times (15)^2 + {}^{n+1}C_3 \times (15)^3 + \ldots + (15)^{n+1} \\ &= (15)^2 \times [{}^{n+1}C_2 + {}^{n+1}C_3 \times 15 + \ldots + (15)^{n-1}] \\ &= 225 \times (\text{an integer}).\end{aligned}$$

Hence, $(2^{4n+4} - 15n - 16)$ is divisible by 225.

EXAMPLE 27 *If a and b are distinct integers then prove that* $(a-b)$ *is a factor of* $(a^n - b^n)$, *whenever n is a positive integer.*

SOLUTION We may write

$$a^n = (a-b+b)^n = \{(a-b)+b\}^n$$

$$= {}^nC_0 \times (a-b)^n + {}^nC_1 \times (a-b)^{(n-1)} \times b + {}^nC_2 \times (a-b)^{(n-2)} \times b^2 + \ldots + {}^nC_{n-1} \times (a-b) \times b^{n-1} + {}^nC_n \times b^n$$

$$= (a-b)^n + nb(a-b)^{n-1} + \frac{1}{2}n(n-1)b^2(a-b)^{n-2} + \ldots + nb^{n-1}(a-b) + b^n$$

$$\Rightarrow \quad (a^n - b^n) = (a-b)^n + nb(a-b)^{n-1} + \frac{1}{2}n(n-1)b^2(a-b)^{n-2} + \ldots + nb^n(a-b)$$

$$= (a-b) \times [(a-b)^{n-1} + nb(a-b)^{n-2} + \frac{1}{2}n(n-1)b^2(a-b)^{n-3} + nb^n]$$

[taking $(a-b)$ common throughout]

$$= (a-b) \times (\text{an integer}).$$

Hence, $(a-b)$ is a factor of $(a^n - b^n)$.

EXAMPLE 28 *The second, third and fourth terms in the binomial expansion of* $(x+a)^n$ *are* 240, 720 *and* 1080 *respectively. Find x, a and n.*

SOLUTION By the binomial expansion, we have

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}a + {}^nC_2 x^{n-2}a^2 + {}^nC_3 x^{n-3}a^3 + \ldots$$

$$\Rightarrow \quad (x+a)^n = x^n + nx^{n-1}a + \frac{1}{2}n(n-1)x^{n-2}a^2 + \frac{1}{6}n(n-1)(n-2)x^{n-3}a^3 + \ldots.$$

$$\therefore \quad T_2 = 240 \Rightarrow nx^{n-1}a = 240, \qquad \ldots \text{(i)}$$

$$T_3 = 720 \Rightarrow n(n-1)x^{n-2}a^2 = 1440, \qquad \ldots \text{(ii)}$$

$$T_4 = 1080 \Rightarrow n(n-1)(n-2)x^{n-3}a^3 = 6480. \qquad \ldots \text{(iii)}$$

On dividing (ii) by (i), we get

$$\frac{(n-1)a}{x} = \frac{1440}{240} \Rightarrow \frac{a}{x} = \frac{6}{(n-1)}. \qquad \ldots \text{(iv)}$$

On dividing (iii) by (ii), we get

$$\frac{(n-2)a}{x} = \frac{6480}{1440} \Rightarrow \frac{a}{x} = \frac{9}{2(n-2)}. \qquad \ldots \text{(v)}$$

Equating the values of $\frac{a}{x}$ from (iv) and (v), we get

$$\frac{6}{(n-1)} = \frac{9}{2(n-2)} \Rightarrow 12(n-2) = 9(n-1) \Rightarrow 3n = 15 \Rightarrow n = 5.$$

Putting $n = 5$ in (i), we get $x^4 a = 48.$ … (vi)

Putting $n = 5$ in (iv), we get $\frac{a}{x} = \frac{6}{4} \Rightarrow \frac{a}{x} = \frac{3}{2} \Rightarrow a = \frac{3}{2}x.$

Putting $a = \frac{3}{2}x$ in (vi), we get

$$x^4 \times \frac{3}{2}x = 48 \Rightarrow x^5 = 48 \times \frac{2}{3} = 32 = 2^5 \Rightarrow x = 2.$$

$$\therefore \quad a = \frac{3}{2} \times 2 = 3.$$

Hence, $x = 2$, $a = 3$, and $n = 5$.

EXAMPLE 29 *If the coefficients of $(r-1)$th, rth and $(r+1)$th terms in the expansion of $(x+1)^n$ are in the ratio 1 : 3 : 5, find the values of n and r.*

SOLUTION We have

$$(x+1)^n = (1+x)^n$$
$$= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_{r-2} x^{r-2} + {}^nC_{r-1} x^{r-1} + {}^nC_r x^r + \ldots .$$

$$\therefore \quad T_{(r-1)} = T_{(r-2)+1} = {}^nC_{r-2} x^{r-2}, \qquad \ldots \text{(i)}$$

$$T_r = T_{(r-1)+1} = {}^nC_{r-1} x^{r-1}, \qquad \ldots \text{(ii)}$$

$$T_{r+1} = {}^nC_r x^r. \qquad \ldots \text{(iii)}$$

It is given that

$$\text{coeff. of } T_{(r-1)} : \text{coeff. of } T_r : \text{coeff. of } T_{(r+1)} = 1 : 3 : 5$$

$$\Rightarrow \quad {}^nC_{(r-2)} : {}^nC_{(r-1)} : {}^nC_r = 1 : 3 : 5$$

$$\Rightarrow \quad \frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}.$$

But, $\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3} \Rightarrow \frac{n!}{(r-2)! \cdot (n-r+2)!} \times \frac{(r-1)! \cdot (n-r+1)!}{n!} = \frac{1}{3}$

$$\Rightarrow \frac{(r-1)}{(n-r+2)} = \frac{1}{3}$$

$$\Rightarrow n - 4r = -5. \qquad \ldots \text{(i)}$$

And, $\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5} \Rightarrow \frac{r}{(n-r+1)} = \frac{3}{5}$

$$\Rightarrow 3n - 8r = -3. \qquad \ldots \text{(ii)}$$

Solving (i) and (ii), we get $n = 7$ and $r = 3$.

Hence, $n = 7$ and $r = 3$.

EXAMPLE 30 *If the coefficients of x^{r-1}, x^r and x^{r+1} in the binomial expansion of $(1+x)^n$ are in AP, prove that*

$$n^2 - n(4r+1) + 4r^2 - 2 = 0.$$

SOLUTION We know that

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_{r-1} x^{r-1} + {}^nC_r x^r + {}^nC_{r+1} x^{r+1} + \ldots .$$

So, the coefficients of x^{r-1}, x^r and x^{r+1} in the given expansion are ${}^nC_{r-1}$, ${}^nC_r$, and ${}^nC_{r+1}$ respectively.

It is given that ${}^nC_{r-1}$, ${}^nC_r$, and ${}^nC_{r+1}$ are in AP.

$\therefore \quad 2\times {}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$

$\Rightarrow \quad \dfrac{{}^nC_{r-1}}{{}^nC_r} + \dfrac{{}^nC_{r+1}}{{}^nC_r} = 2$

$\Rightarrow \quad \dfrac{n!}{(r-1)!\cdot(n-r+1)!}\times\dfrac{(r!)\cdot(n-r)!}{n!}$

$$+\frac{n!}{(r+1)!\cdot(n-r+1)!}\times\frac{(r!)\cdot(n-r)!}{n!}=2$$

$\Rightarrow \quad \dfrac{r}{(n-r+1)} + \dfrac{(n-r)}{(r+1)} = 2$

$[\because\ (n-r+1)! = (n-r+1)\cdot\{(n-r)!\}$ and $r! = r\cdot\{(r-1)!\}]$

$\Rightarrow \quad r(r+1) + (n-r)(n-r+1) = 2(r+1)(n-r+1)$

$\Rightarrow \quad n^2 - n(4r+1) + 4r^2 - 2 = 0.$

Hence, $n^2 - n(4r+1) + 4r^2 - 2 = 0.$

EXAMPLE 31 *Simplify the expression*

$$(1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{998} + \ldots + x^{1000}$$

and find the coefficient of x^{50}.

SOLUTION Clearly, the given series is a geometric series in which

$$a = (1+x)^{1000},\ r = \frac{x(1+x)^{999}}{(1+x)^{1000}} = \frac{x}{(1+x)} \text{ and } n = 1001.$$

$\therefore$ given sum $= \dfrac{a(1-r^n)}{(1-r)}$

$$= \frac{(1+x)^{1000}\times\left\{1-\left(\frac{x}{1+x}\right)^{1001}\right\}}{\left(1-\frac{x}{1+x}\right)} = (1+x)^{1001} - x^{1001}$$

$$= 1 + {}^{1001}C_1 x + {}^{1001}C_2 x^2 + \ldots {}^{1001}C_{1000}x^{1000} + x^{1001} - x^{1001}$$

$$= 1 + {}^{1001}C_1 x + {}^{1001}C_2 x^2 + \ldots + {}^{1001}C_1 x^{1000}.$$

$\therefore$ coefficient of x^{50} in the above expansion

$$= {}^{1001}C_{50} = \frac{(1001)!}{(50)!\cdot(1001-50)!} = \frac{(1001)!}{(50)!\cdot(951)!}.$$

EXAMPLE 32 *If n is a positive integer, find the coefficient of* x^{-1} *in the expansion of* $(1+x)^n\left(1+\dfrac{1}{x}\right)^n$.

SOLUTION We have

$$(1+x)^n\left(1+\frac{1}{x}\right)^n = (1+x)^n \times \frac{(1+x)^n}{x^n} = \frac{(1+x)^{2n}}{x^n}.$$

Coefficient of x^{-1} in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$

$= $ coefficient of x^{-1} in the expansion of $\frac{(1+x)^{2n}}{x^n}$

$= $ coefficient of $(x^{-1}\times x^n)$ in the expansion of $(1+x)^{2n}$

$= $ coefficient of x^{n-1} in the expansion of $(1+x)^{2n}$

$= {}^{2n}C_{n-1}$ $\left[\because\ (1+x)^{2n}=\sum_{r=0}^{2n}{}^{2n}C_r x^r\right]$.

EXAMPLE 33 *Find the coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^{11}$.*

SOLUTION We have

$$(1+x+x^2+x^3)^{11}=\{(1+x)+x^2(1+x)\}^{11}=\{(1+x)(1+x^2)\}^{11}$$

$$=(1+x)^{11}\times(1+x^2)^{11}$$

$$=({}^{11}C_0x^0+{}^{11}C_1x+{}^{11}C_2x^2+{}^{11}C_3x^3+{}^{11}C_4x^4+\ldots)$$

$$\times[{}^{11}C_0(x^2)^0+{}^{11}C_1(x^2)+{}^{11}C_2(x^2)^2+\ldots]$$

$$=(1+11x+55x^2+165x^3+330x^4+\ldots)$$

$$\times(1+11x^2+55x^4+\ldots).$$

$\therefore$ coefficient of x^4 in the given expansion

$$=(1\times55)+(55\times11)+(330\times1)$$

$$=(55+605+330)=990.$$

Hence, the coefficient of x^4 in the given expansion is 990.

EXAMPLE 34 *Find the coefficient of x^4 in the product $(1+2x)^4\times(2-x)^5$.*

SOLUTION Using the binomial expansion, we get

$$(1+2x)^4={}^4C_0+{}^4C_1(2x)+{}^4C_2(2x)^2+{}^4C_3(2x)^3+{}^4C_4(2x)^4$$

$$=1+8x+24x^2+32x^3+16x^4$$

and $(2-x)^5=2^5-{}^5C_1(2^4)x+{}^5C_2(2^3)x^2-{}^5C_3(2^2)x^3$

$$+{}^5C_4(2)x^4-{}^5C_5x^5$$

$$=32-80x+80x^2-40x^3+10x^4-x^5$$

$\therefore$ $(1+2x)^4\times(2-x)^5$

$$=(1+8x+24x^2+32x^3+16x^4)$$

$$\times(32-80x+80x^2-40x^3+10x^4-x^5).$$

Sum of the terms containing x^4 in the given product

$$=(1\times10x^4)+8x(-40x^3)+(24x^2)(80x^2)$$

$$+(32x^3)(-80x)+(16x^4\times32)$$

$$=10x^4-320x^4+1920x^4-2560x^4+512x^2$$

$$=(10-320+1920-2560+512)x^4=-438x^4.$$

Hence, the coefficient of x^4 in the given product is -438.

EXAMPLE 35 *If P be the sum of all odd terms and Q that of all even terms in the expansion of* $(x+a)^n$, *prove that*

(i) $(P^2-Q^2)=(x^2-a^2)^n$,

(ii) $4PQ=\{(x+a)^{2n}-(x-a)^{2n}\}$,

(iii) $2(P^2+Q^2)=[(x+a)^{2n}+(x-a)^{2n}]$.

SOLUTION By the binomial expansion, we have

$$(x+a)^n={}^nC_0x^n+{}^nC_1x^{n-1}+{}^nC_2x^{n-2}+{}^nC_3x^{n-3}+{}^nC_4x^{n-4}+\ldots+{}^nC_na^n.$$

$$=(T_1+T_2+T_3+T_4+T_5+\ldots+T_n+T_{n+1})$$

$$=(T_1+T_3+T_5+\ldots)+(T_2+T_4+T_6+\ldots)$$

$$=P+Q. \quad \ldots (I)$$

And, $(x-a)^n={}^nC_0x^n-{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2-{}^nC_3x^{n-3}a^3+\ldots+(-1)^n\,{}^nC_na^n$

$$=(T_1-T_2+T_3-T_4+\ldots)$$

$$=(T_1+T_3+T_5+\ldots)-(T_2+T_4+T_6+\ldots)$$

$$=P-Q. \quad \ldots (II)$$

(i) On multiplying (I) and (II), we get

$$(x^2-a^2)^n=(P^2-Q^2).$$

(ii) On squaring (I) and (II) and subtracting, we get

$$\{(x+a)^{2n}-(x-a)^{2n}\}=\{(P+Q)^2-(P-Q)^2\}=4PQ.$$

(iii) On squaring (I) and (II) and adding, we get

$$\{(x+a)^{2n}+(x-a)^{2n}\}=\{(P+Q)^2+(P-Q)^2\}=2(P^2+Q^2).$$

EXAMPLE 36 *Using binomial theorem, find the value of* $(103)^4$.

SOLUTION By the binomial expansion, we have

$$(103)^4=(100+3)^4$$

$$={}^4C_0\cdot(100)^4+{}^4C_1\times(100)^3\times3+{}^4C_2\times(100)^2\times3^2+{}^4C_3\times100\times3^3+{}^4C_4\times3^4$$

$$=(100)^4+12\times(100)^3+54\times(100)^2+108\times100+81$$

$$=100000000+12\times1000000+540000+10800+81$$

$$=100000000+12000000+540000+10800+81$$

$$=112550881.$$

Hence, $(103)^4=112550881$.

EXAMPLE 37 *Using binomial theorem, find the value of* $(99)^4$.

SOLUTION By the binomial expansion, we have

$$(99)^4=(100-1)^4$$

$$={}^4C_0\times(100)^4-{}^4C_1\times(100)^3\times1+{}^4C_2\times(100)^2\times1^2-{}^4C_3\times100\times1^3+{}^4C_4\times1^4$$

$$= (100)^4 - 4 \times (100)^3 + 6 \times (100)^2 - 4 \times 100 + 1$$
$$= 100000000 - 4000000 + 60000 - 400 + 1$$
$$= 96059601.$$

Hence, $(99)^4 = 96059601$.

EXAMPLE 38 *Using binomial theorem, find the value of* $(0.99)^{15}$ *up to four places of decimal.*

SOLUTION We have

$$(0.99)^{15} = (1 - 0.01)^{15}$$
$$= 1 - {}^{15}C_1 \times (0.01) + {}^{15}C_2 \times (0.01)^2 - {}^{15}C_3 \times (0.01)^3 + \ldots$$

[neglecting higher powers of 0.01]

$$= 1 - 15 \times (0.01) + 105 \times (0.0001) - 455 \times (0.000001) + \ldots$$
$$= 1 - 0.15 + 0.0105 + 0.000455 = 0.860045.$$

Hence, $(0.99)^{15} = 0.860045$.

EXAMPLE 39 *Using binomial theorem, prove that* $(101)^{50} > (100^{50} + 99^{50})$.

SOLUTION Putting $(101)^{50} = a$ and $(100^{50} + 99^{50}) = b$, we get

$$(a - b) = (101)^{50} - (100)^{50} - (99)^{50}$$
$$= (101)^{50} - (99)^{50} - (100)^{50}$$
$$= (100 + 1)^{50} - (100 - 1)^{50} - (100)^{50}$$
$$= 2 \times [{}^{50}C_1 \times 100^{49} + {}^{50}C_3 \times 100^{47} + \ldots + {}^{50}C_{49} \times 100] - (100)^{50}$$
$$= [2 \times {}^{50}C_3 \times 100^{47} + 2 \times {}^{50}C_5 \times 100^{45} + \ldots + 2 \times {}^{50}C_{49} \times 100]$$
$$= (\text{a positive integer}).$$

Thus, $a - b > 0 \Rightarrow a > b \Rightarrow (101)^{50} > (100^{50} + 99^{50})$.

EXAMPLE 40 *Which is larger,* $(1.01)^{1000000}$ *or* 10000?

SOLUTION We have

$$(1.01)^{1000000} - 10000$$
$$= (1.01)^p - 10000, \text{ where } 1000000 = p$$
$$= (1 + 0.01)^p - 10000$$
$$= {}^pC_0 + {}^pC_1 \times 0.01 + {}^pC_2 \times (0.01)^2 + \ldots + {}^pC_p \times (0.01)^p - 10000$$
$$= 1 + (p \times 0.01) + [{}^pC_2 \times (0.01)^2 + \ldots + (0.01)^p] - 10000$$
$$= 1 + (1000000 \times 0.01) + [{}^{1000000}C_2 \times (0.01)^2 + \ldots + (1.01)^{1000000}]$$
$$- 10000$$
$$= 1 + 10000 + [\text{a positive real number}] - 10000$$
$$= 1 + (\text{a positive real number}) > 0$$

Hence, $(1.01)^{1000000} > 10000$.

EXERCISE 10A

Using binomial theorem, expand each of the following:

1. $(1-2x)^5$
2. $(2x-3)^6$
3. $(3x+2y)^5$
4. $(2x-3y)^4$
5. $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$
6. $\left(x^2-\frac{2}{x}\right)^7$
7. $\left(x-\frac{1}{y}\right)^5$
8. $(\sqrt{x}+\sqrt{y})^8$
9. $(\sqrt[3]{x}-\sqrt[3]{y})^6$
10. $(1+2x-3x^2)^4$
11. $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x\neq 0$
12. $(3x^2-2ax+3a^2)^3$

Evaluate:

13. $(\sqrt{2}+1)^6+(\sqrt{2}-1)^6$
14. $(\sqrt{3}+1)^5-(\sqrt{3}-1)^5$
15. $(2+\sqrt{3})^7+(2-\sqrt{3})^7$
16. $(\sqrt{3}+\sqrt{2})^6-(\sqrt{3}-\sqrt{2})^6$
17. Prove that $\sum_{r=0}^{n} {}^nC_r \cdot 3^r = 4^n$.
18. Using binomial theorem, evaluate each of the following:
    (i) $(101)^4$ (ii) $(98)^4$ (iii) $(1.2)^4$
19. Using binomial theorem, prove that $(2^{3n}-7n-1)$ is divisible by 49, where $n\in N$.
20. Prove that $(2+\sqrt{x})^4+(2-\sqrt{x})^4=2(16+24x+x^2)$.
21. Find the 7th term in the expansion of $\left(\frac{4x}{5}+\frac{5}{2x}\right)^8$.
22. Find the 9th term in the expansion of $\left(\frac{a}{b}-\frac{b}{2a^2}\right)^{12}$.
23. Find the 16th term in the expansion of $(\sqrt{x}-\sqrt{y})^{17}$.
24. Find the 13th term in the expansion of $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}, x\neq 0$.
25. If the coefficients of x^7 and x^8 in the expansion of $\left(2+\frac{x}{3}\right)^n$ are equal then find the value of *n*.
26. Find the ratio of the coefficient of x^{15} to the term independent of *x* in the expansion of $\left(x^2+\frac{2}{x}\right)^{15}$.
27. Show that the ratio of the coefficient of x^{10} in the expansion of $(1-x^2)^{10}$ and the term independent of *x* in the expansion of $\left(x-\frac{2}{x}\right)^{10}$ is 1 : 32.
28. Find the term independent of *x* in the expansion of $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.
29. Find the coefficient of *x* in the expansion of $(1-3x+7x^2)(1-x)^{16}$.

30. Find the coefficient of
    (i) x^5 in the expansion of $(x+3)^8$.
    (ii) x^6 in the expansion of $\left(3x^2-\frac{1}{3x}\right)^9$.
    (iii) x^{-15} in the expansion of $\left(3x^2-\frac{a}{3x^3}\right)^{10}$.
    (iv) a^7b^5 in the expansion of $(a-2b)^{12}$.

31. Show that the term containing x^3 does not exist in the expansion of $\left(3x-\frac{1}{2x}\right)^8$.

32. Show that the expansion of $\left(2x^2-\frac{1}{x}\right)^{20}$ does not contain any term involving x^9.

33. Show that the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$ does not contain any term involving x^{-1}.

34. Write the general term in the expansion of $(x^2-y)^6$.

35. Find the 5th term from the end in the expansion of $\left(x-\frac{1}{x}\right)^{12}$.

36. Find the 4th term from the end in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$.

37. If the 7th terms from the beginning and end in the expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$ are equal, find the value of *n*.

38. Find the middle term in the expansion of:
    (i) $(3+x)^6$
    (ii) $\left(\frac{x}{3}+3y\right)^8$
    (iii) $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$
    (iv) $\left(x^2-\frac{2}{x}\right)^{10}$

39. Find the two middle terms in the expansion of:
    (i) $(x^2+a^2)^5$
    (ii) $\left(x^4-\frac{1}{x^3}\right)^{11}$
    (iii) $\left(\frac{p}{x}+\frac{x}{p}\right)^9$
    (iv) $\left(3x-\frac{x^3}{6}\right)^9$

40. Find the term independent of *x* in the expansion of:
    (i) $\left(2x+\frac{1}{3x^2}\right)^9$
    (ii) $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^6$
    (iii) $\left(x-\frac{1}{x^2}\right)^{3n}$
    (iv) $\left(3x-\frac{2}{x^2}\right)^{15}$

41. Find the coefficient of x^5 in the expansion of $(1+x)^3(1-x)^6$.

42. Find numerically the greatest term in the expansion of $(2+3x)^9$, where

$x = \frac{3}{2}$.

43. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^{2n}$ are in AP, show that $2n^2 - 9n + 7 = 0$.

44. Find the 6th term of the expansion $(y^{1/2} + x^{1/3})^n$, if the binomial coefficient of 3rd term from the end is 45.

45. If the 17th and 18th terms in the expansion of $(2+a)^{50}$ are equal, find the value of a.

46. Find the coefficient of x^4 in the expansion of $(1+x)^n(1-x)^n$. Deduce that $C_2 = C_0C_4 - C_1C_3 + C_2C_2 - C_3C_1 + C_4C_0$, where C_r stands for ${}^nC_r$.

47. Prove that the coefficient of x^n in the binomial expansion of $(1+x)^{2n}$ is twice the coefficient of x^n in the binomial expansion of $(1+x)^{2n-1}$.

48. If the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ is 1120, find p.

ANSWERS (EXERCISE 10A)

1. $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

2. $64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

3. $243x^5 + 810x^4y + 1080x^3y^2 + 720x^2y^3 + 240xy^4 + 32y^5$

4. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

5. $\frac{64x^6}{729} - \frac{32x^4}{27} + \frac{20x^2}{3} - 20 + \frac{135}{4x^2} - \frac{243}{8x^4} + \frac{729}{64x^6}$

6. $x^{14} - 14x^{11} + 84x^8 - 280x^5 + 560x^2 - \frac{672}{x} + \frac{448}{x^4} - \frac{128}{x^7}$

7. $x^5 - \frac{5x^4}{y} + \frac{10x^3}{y^2} - \frac{10x^2}{y^3} + \frac{5x}{y^4} - \frac{1}{y^5}$

8. $x^4 + 8x^{7/2}y^{1/2} + 28x^3y + 56x^{5/2}y^{3/2} + 70x^2y^2 + 56x^{3/2}y^{5/2} + 28xy^3 + 8x^{1/2}y^{7/2} + y^4$

9. $x^2 - 6x^{5/3}y^{1/3} + 15x^{4/3}y^{2/3} - 20xy + 15x^{2/3}y^{4/3} - 6x^{1/3}y^{5/3} + y^2$

10. $81x^8 - 216x^7 + 108x^6 + 120x^5 - 74x^4 - 40x^3 + 12x^2 + 8x + 1$

11. $\frac{x^4}{16} + \frac{x^3}{2} + \frac{x^2}{2} - 4x - 5 + \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4}$

12. $27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$

13. 198 **14.** 152 **15.** 10084 **16.** $396\sqrt{6}$

18. (i) 104060401 (ii) 92236816 (iii) 2.0736

21. $\frac{4375}{x^4}$ **22.** $\frac{495b^4}{256a^{12}}$ **23.** $-136xy^{15/2}$ **24.** 18564

25. $n = 56$ **26.** 1 : 32 **28.** $\frac{17}{54}$ **29.** –19

30. (i) 1512 (ii) 378 (iii) $\frac{-40}{27}a^7$ (iv) -25344

34. $T_{r+1} = (-1)^r \times {}^6C_r \times x^{(12-2r)} \times y^r$ **35.** $495x^4$ **36.** $\frac{10500}{x^3}$

37. $n = 12$ **38.** (i) $540x^3$ (ii) $70x^4y^4$ (iii) -252 (iv) $-8064x^5$

39. (i) $10x^6a^4, 10x^4a^6$ (ii) $-462x^9, 462x^2$ (iii) $\frac{126p}{x}, \frac{126x}{p}$ (iv) $\frac{189x^{17}}{8}, \frac{-21x^{19}}{16}$

40. (i) $\frac{1792}{9}$ (ii) $\frac{5}{12}$ (iii) $(-1)^n \cdot {}^{3n}C_n$ (iv) $-3003 \times 3^{10} \times 2^5$

41. -6 **42.** $T_7 = \frac{7 \times 3^{13}}{2}$ **44.** $252y^{5/2}x^{5/3}$ **45.** $a = 1$

46. C_2 **48.** $p = \pm 2$

HINTS TO SOME SELECTED QUESTIONS

8. $(x^{1/2} + y^{1/2})^8$

$= {}^8C_0(x^{1/2})^8 + {}^8C_1(x^{1/2})^7(y^{1/2}) + {}^8C_2(x^{1/2})^6(y^{1/2})^2 + {}^8C_3(x^{1/2})^5(y^{1/2})^3$
$+ {}^8C_4(x^{1/2})^4(y^{1/2})^4 + {}^8C_5(x^{1/2})^3(y^{1/2})^5 + {}^8C_6(x^{1/2})^2(y^{1/2})^6 + {}^8C_7(x^{1/2})(y^{1/2})^7$
$+ {}^8C_8(y^{1/2})^8$

$= x^4 + 8x^{7/2}y^{1/2} + 28x^3y + 56x^{5/2}y^{3/2} + 70x^2y^2 + 56x^{3/2}y^{5/2} + 28xy^3$
$+ 8x^{1/2}y^{7/2} + y^4.$

9. $(x^{1/3} - y^{1/3})^6$

$= {}^6C_0(x^{1/3})^6 - {}^6C_1(x^{1/3})^5(y^{1/3}) + {}^6C_2(x^{1/3})^4(y^{1/3})^2 - {}^6C_3(x^{1/3})^3(y^{1/3})^3$
$+ {}^6C_4(x^{1/3})^2(y^{1/3})^4 - {}^6C_5(x^{1/3})(y^{1/3})^5 + {}^6C_6(y^{1/3})^6$

$= x^2 - 6x^{5/3}y^{1/3} + 15x^{4/3}y^{2/3} - 20xy + 15x^{2/3}y^{4/3} - 6x^{1/3}y^{5/3} + y^2.$

10. $(1 + 2x - 3x^2)^4 = [1 + (2x - 3x^2)]^4$

$= (1 + y)^4$, where $2x - 3x^2 = y$

$= {}^4C_0 + {}^4C_1y + {}^4C_2y^2 + {}^4C_3y^3 + {}^4C_4y^4$

$= 1 + 4y + 6y^2 + 4y^3 + y^4$

$= 1 + 4(2x - 3x^2) + 6(2x - 3x^2)^2 + 4(2x - 3x^2)^3 + (2x - 3x^2)^4$

$= 1 + (8x - 12x^2) + 6[{}^2C_0(2x)^2 - {}^2C_1(2x)(3x^2) + {}^2C_2(3x^2)^2]$
$+ 4[{}^3C_0(2x)^3 - {}^3C_1(2x)^2(3x^2) + {}^3C_2(2x)(3x^2)^2 - {}^3C_3(3x^2)^3]$
$+ [{}^4C_0(2x)^4 - {}^4C_1(2x)^3(3x^2) + {}^4C_2(2x)^2(3x^2)^2 - {}^4C_3(2x)(3x^2)^3 + {}^4C_4(3x^2)^4]$

$= 1 + (8x - 12x^2) + 6(4x^2 - 12x^3 + 9x^4) + 4(8x^3 - 36x^4 + 54x^5 - 27x^6)$
$+ (16x^4 - 96x^5 + 216x^6 - 216x^7 + 81x^8)$

$= 1 + (8x - 12x^2) + (24x^2 - 72x^3 + 54x^4) + (32x^3 - 144x^4 + 216x^5 - 108x^6)$
$+ (16x^4 - 96x^5 + 216x^6 - 216x^7 + 81x^8)$

$= 81x^8 - 216x^7 + 108x^6 + 120x^5 - 74x^4 - 40x^3 + 12x^2 + 8x + 1.$

11. $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4 = \left\{1 + \left(\frac{x}{2} - \frac{2}{x}\right)\right\}^4$

$= (1 + y)^4$, where $\left(\frac{x}{2} - \frac{2}{x}\right) = y$

$$= {}^4C_0 + {}^4C_1 y + {}^4C_2 y^2 + {}^4C_3 y^3 + {}^4C_4 y^4$$
$$= 1 + 4y + 6y^2 + 4y^3 + y^4$$
$$= 1 + 4\left(\frac{x}{2} - \frac{2}{x}\right) + 6\left(\frac{x}{2} - \frac{2}{x}\right)^2 + 4\left(\frac{x}{2} - \frac{2}{x}\right)^3 + \left(\frac{x}{2} - \frac{2}{x}\right)^4$$
$$= 1 + \left(2x - \frac{8}{x}\right) + 6\left(\frac{x^2}{4} + \frac{4}{x^2} - 2\right) + 4\left[{}^3C_0\left(\frac{x}{2}\right)^3 - {}^3C_1\left(\frac{x}{2}\right)^2\left(\frac{2}{x}\right) + {}^3C_2\left(\frac{x}{2}\right)\left(\frac{2}{x}\right)^2 - {}^3C_3\left(\frac{2}{x}\right)^3\right]$$
$$+\left[{}^4C_0\left(\frac{x}{2}\right)^4 - {}^4C_1\left(\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + {}^4C_2\left(\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^4C_3\left(\frac{x}{2}\right)\left(\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{2}{x}\right)^4\right]$$
$$= 1 + \left(2x - \frac{8}{x}\right) + \left(\frac{3x^2}{2} + \frac{24}{x^2} - 12\right) + 4\left[\frac{x^3}{8} - \frac{3}{2}x + \frac{6}{x} - \frac{8}{x^3}\right] + \left[\frac{x^4}{16} - x^2 + 6 - \frac{16}{x^2} + \frac{16}{x^4}\right]$$
$$= 1 + \left(2x - \frac{8}{x}\right) + \left(\frac{3x^2}{2} + \frac{24}{x^2} - 12\right) + \left(\frac{x^3}{2} - 6x + \frac{24}{x} - \frac{32}{x^3}\right) + \left(\frac{x^4}{16} - x^2 + 6 - \frac{16}{x^2} + \frac{16}{x^4}\right)$$
$$= \frac{x^4}{16} + \frac{x^3}{2} + \frac{x^2}{2} - 4x - 5 + \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4}.$$

12. $(3x^2 - 2ax + 3a^2)^3 = [(3x^2 - 2ax) + 3a^2]^3$
$= {}^3C_0(3x^2 - 2ax)^3 + {}^3C_1(3x^2 - 2ax)^2(3a^2) + {}^3C_2(3x^2 - 2ax)(3a^2)^2 + {}^3C_3(3a^2)^3$
$= [{}^3C_0(3x^2)^3 - {}^3C_1(3x^2)^2 \times 2ax + {}^3C_2(3x^2) \times (2ax)^2 - {}^3C_3(2ax)^3]$
$+3(9x^4 + 4a^2x^2 - 12ax^3)(3a^2) + (27a^4)(3x^2 - 2ax) + 27a^6$
$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 + 36a^4x^2 - 108a^3x^3 + 81a^4x^2 - 54a^5x + 27a^6$
$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6.$

13. $(a+b)^6 + (a-b)^6 = 2[a^6 + {}^6C_2a^4b^2 + {}^6C_4a^2b^4 + {}^6C_6b^6].$
$\therefore \quad (\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 2[(\sqrt{2})^6 + {}^6C_2(\sqrt{2})^4 \times 1^2 + {}^6C_4(\sqrt{2})^2 \times 1^4 + {}^6C_6(1)^6]$
$= 2(8 + 60 + 30 + 1) = 2(99) = 198.$

16. $(a+b)^6 - (a-b)^6 = 2[{}^6C_1a^5b + {}^6C_3a^3b^3 + {}^6C_5ab^5].$
$\therefore \quad (\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6 = 2[6(\sqrt{3})^5(\sqrt{2}) + 20(\sqrt{3})^3(\sqrt{2})^3 + 6(\sqrt{3})(\sqrt{2})^5]$
$= 2[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}] = 396\sqrt{6}.$

17. LHS $= {}^nC_0 \times 3^0 + {}^nC_1 \times 3 + {}^nC_2 \times 3^2 + \ldots + {}^nC_n 3^n$
$= (1+3)^n = 4^n =$ RHS [by binomial expansion].

18. (i) $(101)^4 = (100+1)^4$
$= {}^4C_0(100)^4 + {}^4C_1(100)^3 \times 1 + {}^4C_2(100)^2 \times 1^2 + {}^4C_3(100) \times 1^3 + {}^4C_4 \times 1^4$
$= 100000000 + 4000000 + 60000 + 400 + 1 = 104060401.$

19. $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n.$
$(1+7)^n = {}^nC_0 + {}^nC_1 \times 7 + {}^nC_2 \times (7)^2 + \ldots + {}^nC_n \times (7)^n$
$= 1 + 7n + (7)^2[{}^nC_2 + {}^nC_3 \times 7 + \ldots + (7)^{n-2}]$
$\Rightarrow \quad (8^n - 7n - 1) = (7)^2 \times (\text{an integer}) = 49 \times (\text{an integer}).$
Hence, $(2^{3n} - 7n - 1)$ is divisible by 49.

20. $(a+b)^4 + (a-b)^4 = [{}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4]$
$+[{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4]$
$= 2[{}^4C_0a^4 + {}^4C_2a^2b^2 + {}^4C_4b^4] = 2[a^4 + 6a^2b^2 + b^4].$

Putting $a = 2$ and $b = \sqrt{x}$, we get

$$(2+\sqrt{x})^4 + (2-\sqrt{x})^4 = 2[2^4 + 6\times 2^2 \times (\sqrt{x})^2 + (\sqrt{x})^4] = 2(16 + 24x + x^2).$$

25. We have, $T_{r+1} = {}^nC_r 2^{n-r} \times \left(\frac{x}{3}\right)^r = {}^nC_r \times 2^{n-r} \times 3^{-r} \times x^r.$

Coeff. of x^7 = coeff. of x^8 $\Rightarrow$ ${}^nC_7 \times 2^{n-7} \times 3^{-7} = {}^nC_8 \times 2^{n-8} \times 3^{-8}$.

$$\therefore \quad \frac{{}^nC_7}{{}^nC_8} = \frac{2^{n-8}\times 3^{-8}}{2^{n-7}\times 3^{-7}} = \frac{1}{2\times 3} = \frac{1}{6} \Rightarrow \frac{n!}{(7!)(n-7)!} \times \frac{(8!)\times(n-8)!}{(n!)} = \frac{1}{6}.$$

$$\therefore \quad \frac{8}{n-8} = \frac{1}{6} \Rightarrow n - 8 = 48 \Rightarrow n = 56.$$

26. $T_{r+1} = {}^{15}C_r (x^2)^{15-r} \times \left(\frac{2}{x}\right)^r = {}^{15}C_r \times 2^r \times x^{30-3r}$

$[30 - 3r = 15 \Rightarrow 3r = 15 \Rightarrow r = 5]$ and $[30 - 3r = 0 \Rightarrow 3r = 30 \Rightarrow r = 10]$.

Find $(T_6 : T_{11}) = (T_{5+1} : T_{10+1})$.

28. In the expansion of $E = \left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$, we have

$$T_{r+1} = (-1)^r \cdot {}^9C_r \cdot \left(\frac{3}{2}x^2\right)^{(9-r)} \left(\frac{1}{3x}\right)^r$$

$$\Rightarrow \quad T_{r+1} = (-1)^r \cdot {}^9C_r \cdot \frac{3^{(9-2r)}}{2^{(9-r)}} \cdot x^{(18-3r)}.$$

$$\therefore \quad (1 + x + 2x^3)\left[\left(a_0 \times \frac{1}{x^3} + a_1 \times \frac{1}{x} + a_2\right) \text{ from } E\right]$$

$$= (1 + x + 2x^3)\left[\left\{(-1)^7 \cdot {}^9C_7 \cdot \frac{3^{-5}}{2^2} \times \frac{1}{x^3}\right\} + \left\{(-1)^6 \cdot {}^9C_6 \cdot \frac{3^{-3}}{2^3} \times x^0\right\}\right]$$

$$\left[x = -1 \Rightarrow 18 - 3r = -1 \Rightarrow r \text{ is fraction}, \quad 18 - 3r = 0 \Rightarrow r = 6 \text{ and } 18 - 3r = -3 \Rightarrow r = 7\right]$$

$$= (1 + x + 2x^3)\left[\frac{-1}{27x^3} + \frac{7}{18}\right]$$

$$\therefore \quad \text{required term} = \left(\frac{-2}{27} + \frac{7}{18}\right) = \frac{17}{54}.$$

29. In the expansion of $E = (1 - x)^{16}$, we have

$$T_{r+1} = (-1)^r \cdot {}^{16}C_r x^r.$$

Product of given expressions $= (1 - 3x + 7x^2)(1 - 16x + \ldots)$.

Terms containing $x = [1 \times (-16x) + (-3x) \times 1] = (-19x)$.

$\therefore$ coefficient of $x = -19$.

30. (iii) $T_{r+1} = (-1)^r \times {}^{10}C_r (3x^2)^{(10-r)} \times \left(\frac{a}{3x^3}\right)^r$

$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^{10}C_r \times 3^{(10-2r)} \times x^{(20-5r)} \times a^r.$$

Now, $20 - 5r = -15 \Rightarrow 5r = 35 \Rightarrow r = 7$.

(iv) $T_{r+1} = (-1)^r \times {}^{12}C_r \times a^{(12-r)} \times (2b)^r$

$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^{12}C_r \times 2^r \times a^{(12-r)} b^r.$$

Put $r = 5$ and get T_6.

31. $T_{r+1} = (-1)^r \cdot {}^8C_r \cdot (3x)^{(8-r)} \cdot \left(\frac{1}{2x}\right)^r$

$$\Rightarrow \quad T_{r+1} = (-1)^r \cdot {}^8C_r \cdot 3^{(8-r)} \cdot \frac{1}{2^r} \cdot x^{(8-2r)}.$$

Now, $8-2r=3 \Rightarrow r=\frac{5}{2}$, which is a fraction.

34. $T_{r+1}=(-1)^r \times {}^6C_r \times x^{(12-2r)} \times y^r$.

35. pth term from the end $=(n-p+2)$th term from the beginning.

$\therefore$ 5th term from the end $=(12-5+2)$th = 9th term.

37. 7th term from the end $=(n-7+2)$th term $=(n-5)$th term.

$\therefore \quad T_7=T_{n-5} \Rightarrow n-5=7 \Rightarrow n=12.$

40. (iii) $T_{r+1}=(-1)^r \cdot {}^{3n}C_r x^{(3n-r)} \times \left(\frac{1}{x^2}\right)^r$

$\Rightarrow \quad T_{r+1}=(-1)^r \cdot {}^{3n}C_r x^{(3n-3r)}$.

Put $3n-3r=0 \Rightarrow r=n$.

So, T_{n+1} is free from x.

And, $T_{n+1}=(-1)^n \cdot {}^{3n}C_n$.

41. $(1+x)^3(1-x)^6=({}^3C_0+{}^3C_1x+{}^3C_2x^2+{}^3C_3x^3)$

$\times({}^6C_0-{}^6C_1x+{}^6C_2x^2-{}^6C_3x^3+{}^6C_4x^4-{}^6C_5x^5+{}^6C_6x^6)$

$=(1+3x+3x^2+x^3)\times(1-6x+15x^2-20x^3+15x^4-6x^5+x^6)$.

Coeff. of x^5 in the given product $=(-6+45-60+15)=-6$.

42. We have, $(2+3x)^9=2^9\left(1+\frac{3x}{2}\right)^9$.

$T_{r+1}=2^9 \times {}^9C_r\left(\frac{3x}{2}\right)^r$ and $T_r=2^9 \times {}^9C_{r-1}\left(\frac{3x}{2}\right)^{r-1}$.

$\therefore \quad \frac{T_{r+1}}{T_r}=2^9 \times {}^9C_r \times \frac{3^r}{2^r} \times x^r \times \frac{2^{r-1}}{3^{r-1}} \times \frac{1}{x^{r-1}} \times \frac{1}{2^9 \times {}^9C_{r-1}}$

$=\frac{3}{2} \times x \times \frac{{}^9C_r}{{}^9C_{r-1}}=\left(\frac{3}{2}x\right) \times \frac{(9!)}{(r!)\cdot(9-r)!} \times \frac{(r-1)!}{(9!)} \times (10-r)!$

$=\left(\frac{3}{2} \times \frac{3}{2}\right) \times \frac{(10-r)}{r}=\frac{9(10-r)}{4r} \qquad \left[\because x=\frac{3}{2}\right]$.

Let the greatest term be T_{r+1}.

Then, $T_{r+1} \geq T_r \Rightarrow \frac{T_{r+1}}{T_r} \geq 1 \Rightarrow \frac{9(10-r)}{4r} \geq 1$

$\Rightarrow 4r \leq 90-9r \Rightarrow 13r \leq 90 \Rightarrow r \leq 6\frac{12}{13}$.

Thus, the maximum value of r is 6 for which T_{r+1} is greatest.

$\therefore$ greatest term $=T_{6+1}=2^9 \times {}^9C_6 \times \frac{3^6}{2^6} \times x^6$, where $x=\frac{3}{2}$.

$\therefore$ numerical value of greatest term at $x=\frac{3}{2}=2^9 \times {}^9C_3 \times \frac{3^6}{2^6} \times \frac{3^6}{2^6}$

$=\frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times \frac{3^{12}}{2^3}=\left(\frac{7 \times 3^{13}}{2}\right)$.

43. It is given that ${}^{2n}C_1$, ${}^{2n}C_2$ and ${}^{2n}C_3$ are in AP.

$\therefore \quad 2 \times {}^{2n}C_2={}^{2n}C_1+{}^{2n}C_3$

$$\Rightarrow \quad 2 = \frac{{}^{2n}C_1}{{}^{2n}C_2} + \frac{{}^{2n}C_3}{{}^{2n}C_2}$$

$$\Rightarrow \quad 2 = \frac{4n}{2n(2n-1)} + \frac{2n-3+1}{3} \qquad \left[\because \frac{{}^{2n}C_r}{{}^{2n}C_{r-1}} = \frac{2n-r+1}{r} \text{ and } r = 3\right]$$

$$\Rightarrow \quad \frac{2}{2n-1} + \frac{2n-2}{3} = 2$$

$$\Rightarrow \quad 6 + (2n-1)(2n-2) = 6(2n-1)$$

$$\Rightarrow \quad 4n^2 - 18n + 14 = 0 \Rightarrow 2n^2 - 9n + 7 = 0.$$

44. Coefficient of 3rd term from the beginning is always equal to the coefficient of 3rd term from the end.

Now, $T_{r+1} = {}^nC_r (y^{1/2})^{n-r} \cdot (x^{1/3})^r$.

Coefficient of $T_3 = {}^nC_2 = \frac{n(n-1)}{2}$.

$$\therefore \quad \frac{n(n-1)}{2} = 45 \Rightarrow n^2 - n - 90 = 0 \Rightarrow n^2 - 10n + 9n - 90 = 0$$

$$\Rightarrow (n-10)(n+9) = 0 \Rightarrow n = 10 \qquad [\because n \neq -9].$$

$$\therefore \quad T_6 = T_{5+1} = {}^{10}C_5 (y^{1/2})^{(10-5)} \cdot (x^{1/3})^5$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} \times y^{5/2} x^{5/3} = 252 y^{5/2} x^{5/3}.$$

45. $T_{r+1} = {}^{50}C_r \times 2^{(50-r)} \times a^r$.

Now, $T_{17} = T_{18} \Rightarrow T_{16+1} = T_{17+1}$

$$\Rightarrow {}^{50}C_{16} \times 2^{34} \times a^{16} = {}^{50}C_{17} \times 2^{33} \times a^{17}$$

$$\Rightarrow a = \frac{2 \times {}^{50}C_{16}}{{}^{50}C_{17}} = \frac{2 \times 17}{(50-17+1)} = 1. \qquad \left[\because \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{r}{n-r+1}\right]$$

46. $(1+x)^n \times (1-x)^n = [C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n] \times [C_0 - C_1 x + C_2 x^2 - \ldots + (-1)^n C_n x^n]$

$(1+x)^n \times (1-x)^n = (1-x^2)^n = [C_0 - C_1 x^2 + C_2 x^4 - \ldots + (-1)^n C_n x^{2n}]$.

$\therefore \quad C_0 C_4 - C_1 C_3 + C_2 C_2 - C_3 C_1 + C_4 C_0 = C_2$.

47. In $(1+x)^{2n}$, we have $T_{r+1} = {}^{2n}C_r \cdot x^r$.

$\therefore$ coefficient of x^n in $(1+x)^{2n} = {}^{2n}C_n = \frac{(2n)!}{(n!)(n!)} = \frac{(2n)(2n-1)!}{n(n-1)! \times (n!)}$

$$= 2 \times \frac{(2n-1)!}{(n-1)! \times (n!)}.$$

In $(1+x)^{2n-1}$, we have $T_{r+1} = {}^{2n-1}C_r x^r$.

$\therefore$ coeff. of x^n in $(1+x)^{2n-1} = {}^{2n-1}C_n = \frac{(2n-1)!}{(n-1)! \times n!}$.

Hence, coeff. of x^n in $(1+x)^{2n} = 2 \times$ coeff. of x^n in $(1+x)^{2n-1}$.

48. T_{r+1} in $\left(\frac{p}{2} + 2\right)^8 = {}^8C_r \left(\frac{p}{2}\right)^{8-r} \cdot 2^r$.

Middle term $= \left(\frac{8}{2} + 1\right)$th = 5th term $= {}^8C_4 \left(\frac{p}{2}\right)^4 \times 2^4$.

$$\therefore \quad \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} \times p^4 = 1120 \Rightarrow p^4 = 16 \Rightarrow p = \pm 2.$$

EXERCISE 10B

Very-Short-Answer Questions

1. Show that the term independent of x in the expansion of $\left(x-\frac{1}{x}\right)^{10}$ is -252.
2. If the coefficients of x^2 and x^3 in the expansion of $(3+px)^9$ are the same then prove that $p=\frac{9}{7}$.
3. Show that the coefficient of x^{-3} in the expansion of $\left(x-\frac{1}{x}\right)^{11}$ is -330.
4. Show that the middle term in the expansion of $\left(\frac{2x^2}{3}+\frac{3}{2x^2}\right)^{10}$ is 252.
5. Show that the coefficient of x^4 in the expansion of $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}$ is $\frac{405}{256}$.
6. Prove that there is no term involving x^6 in the expansion of $\left(2x^2-\frac{3}{x}\right)^{11}$.
7. Show that the coefficient of x^4 in the expansion of $(1+2x+x^2)^5$ is 212.
8. Write the number of terms in the expansion of $(\sqrt{2}+1)^5+(\sqrt{2}-1)^5$.
9. Which term is independent of x in the expansion of $\left(x-\frac{1}{3x^2}\right)^9$?
10. Write the coefficient of the middle term in the expansion of $(1+x)^{2n}$.
11. Write the coefficient of x^7y^2 in the expansion of $(x+2y)^9$.
12. If the coefficients of $(r-5)$th and $(2r-1)$th terms in the expansion of $(1+x)^{34}$ are equal, find the value of r.
13. Write the 4th term from the end in the expansion of $\left(\frac{3}{x^2}-\frac{x^3}{6}\right)^7$.
14. Find the coefficient of x^n in the expansion of $(1+x)(1-x)^n$.
15. In the binomial expansion of $(a+b)^n$, the coefficients of the 4th and 13th terms are equal to each other. Find the value of n.
16. Find the positive value of m for which the coefficient of x^2 in the expansion of $(1+x)^m$ is 6.

ANSWERS (EXERCISE 10B)

8. Three 9. 4th 10. ${}^{2n}C_n$ 11. 144 12. 14
13. $\frac{35x^6}{48}$ 14. $(-1)^n\times(n-1)$ 15. $n=15$ 16. $m=4$

HINTS TO SOME SELECTED QUESTIONS

1. $T_{r+1} = (-1)^r \cdot {}^{10}C_r x^{(10-r)} \cdot \left(\frac{1}{x}\right)^r = (-1)^r \cdot {}^{10}C_r x^{(10-2r)}$.

   Putting $10 - 2r = 0$, we get $r = 5$.

   $\therefore \quad T_6 = (-1)^5 \cdot {}^{10}C_5 = -\frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = -252.$

2. $T_{r+1} = {}^9C_r 3^{(9-r)} \times (px)^r = {}^9C_r 3^{(9-r)} \cdot p^r x^r$.

   Coeff. of x^2 = coeff. of $x^3 \Rightarrow {}^9C_2 \times 3^7 \times p^2 = {}^9C_3 \times 3^6 \times p^3$.

3. $T_{r+1} = (-1)^r \cdot {}^{11}C_r x^{(11-r)} \cdot \left(\frac{1}{x}\right)^r = (-1)^r \cdot {}^{11}C_r x^{(11-2r)}$.

   Now, $11 - 2r = -3 \Rightarrow 2r = 14 \Rightarrow r = 7$.

   $T_8 = (-1)^7 \cdot {}^{11}C_7 x^{-3} = -{}^{11}C_4 x^{-3} = -\frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} x^{-3} = -330x^{-3}$.

4. $T_{r+1} = {}^{10}C_r \left(\frac{2x^2}{3}\right)^{10-r} \times \left(\frac{3}{2x^2}\right)^r = {}^{10}C_r \times 2^{10-2r} \times 3^{2r-10} \times x^{(20-4r)}$.

   The expansion has 11 terms. So, middle term = 6th term.

   $T_6 = T_{5+1} = {}^{10}C_5 \times 2^0 \times 3^0 \times x^0 = \left(\frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}\right) = 252.$

5. $T_{r+1} = (-1)^r \cdot {}^{10}C_r \cdot \left(\frac{x}{2}\right)^{(10-r)} \times \left(\frac{3}{x^2}\right)^r = (-1)^r \cdot {}^{10}C_r \cdot 3^r \cdot 2^{r-10} \cdot x^{(10-3r)}$.

   Putting $10 - 3r = 4$, we get $3r = 6 \Rightarrow r = 2$.

   $$T_3 = T_{(2+1)} = (-1)^2 \cdot {}^{10}C_2 \times 3^2 \times 2^{-8} \times x^4 = \frac{405}{256} x^4.$$

6. $T_{r+1} = (-1)^r \cdot {}^{11}C_r (2x^2)^{(11-r)} \cdot \left(\frac{3}{x}\right)^r = (-1)^r \cdot {}^{11}C_r 2^{(11-r)} \times 3^r \cdot x^{22-3r}$

   $22 - 3r = 6 \Rightarrow 3r = 16 \Rightarrow r = \frac{16}{3}$, which is a fraction.

7. $(1 + 2x + x^2)^5 = \{(1+x)^2\}^5 = (1+x)^{10}$.

   $T_{r+1} = {}^{10}C_r x^r$. Coefficient of $x^4 = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.$

8. Each expansion gives 6 terms. Their sum will contain 3 terms, as 3 terms are cancelled.

9. $T_{r+1} = (-1)^r \cdot {}^9C_r \cdot x^{9-r} \cdot \left(\frac{1}{3x^2}\right)^r = (-1)^r \cdot {}^9C_r \cdot x^{(9-3r)} \cdot \frac{1}{3^r}$.

   Putting $9 - 3r = 0$, we get $r = 3$.

   Hence, T_4, i.e., 4th term is independent of x.

10. Middle term $= \left(\frac{2n}{2} + 1\right)$th term $= (n+1)$th term $= T_{n+1}$.

    Now, $T_{r+1} = {}^{2n}C_r x^r \Rightarrow T_{n+1} = {}^{2n}C_n x^n$. Its coefficient is ${}^{2n}C_n$.

11. $T_{r+1} = {}^9C_r x^{(9-r)} \cdot (2y)^r = {}^9C_r \times 2^r \times x^{(9-r)} \times y^r$.

    Putting $r = 2$, we get the coefficient of $x^7 y^2$ as ${}^9C_2 \times 2^2 = 144$.

12. $T_m = T_{(m-1)+1} = {}^{34}C_{m-1} \times x^{m-1}$.

    $\therefore \quad T_{r-5} = {}^{34}C_{r-6} x^{r-6}$ and $T_{2r-1} = {}^{34}C_{2r-2} x^{2r-2}$.

    Now, ${}^{34}C_{r-6} = {}^{34}C_{2r-2} \Rightarrow (r - 6 = 2r - 2)$ or $(r - 6 + 2r - 2 = 34)$

    $\Rightarrow r = -4$ or $r = 14 \Rightarrow r = 14$.

13. rth term from the end $= (n - r + 2)$th term.

    $\therefore \quad$ 4th term from the end $= (7 - 4 + 2)$th term = 5th term.

$T_{r+1} = (-1)^r \cdot {}^7C_r \left(\frac{3}{4x^2}\right)^{(7-r)} \cdot \left(\frac{x^3}{6}\right)^r = (-1)^r \times {}^7C_r \times 3^{(7-2r)} \times 2^{-r} \times x^{5r-14}.$

$T_5 = T_{(4+1)} = (-1)^4 \times {}^7C_4 \times 3^{-1} \times 2^{-4} \times x^6 = {}^7C_3 \times \frac{1}{(3 \times 2^4)} x^6 = \frac{35x^6}{48}.$

14. T_{r+1} in $(1-x)^n = (-1)^{r+1} \cdot {}^nC_r \cdot x^r.$

$(1+x)(1-x)^n = (1+x)[1 - {}^nC_1 x + \ldots + (-1)^n \times {}^nC_{n-1} x^{n-1} + (-1)^{n+1} \cdot {}^nC_n x^n].$

Coeff. of x^n in the given product $(1+x)(1-x)^n = [(-1)^{n+1} \cdot {}^nC_n + (-1)^n \cdot {}^nC_{n-1}]$

$= (-1)^{n+1} + (-1)^n \times n = (-1)^n \times (n-1).$

15. $T_{r+1} = {}^nC_r a^{n-r} \cdot b^r.$

Coeff. of T_4 = coeff. of $T_{13} \Rightarrow {}^nC_3 = {}^nC_{12} \Rightarrow (3+12) = n \Rightarrow n = 15.$

16. $(1+x)^m = {}^mC_0 + {}^mC_1 x + {}^mC_2 x^2 + \ldots + {}^mC_m x^m.$

$\therefore \quad {}^mC_2 = 6 \Rightarrow \frac{m(m-1)}{2!} = 6 \Rightarrow m^2 - m - 12 = 0 \Rightarrow (m-4)(m+3) = 0 \Rightarrow m = 4.$

REVIEW OF FACTS AND FORMULAE

1. Binomial Theorem

If *x* and *a* are real numbers then for all $n \in N$, we have

$$(x+a)^n = {}^nC_0 x^n a^0 + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + {}^nC_r x^{n-r} a^r + \ldots + {}^nC_n x^0 a^n$$

$$\Rightarrow \quad (x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r.$$

In this expansion, we have

(i) total number of terms = $(n+1)$.

(ii) sum of the powers of *x* and *a* in each term is *n*.

(iii) coefficients of terms equidistant from the beginning and the end are equal.

(iv) general term is given by $T_{r+1} = {}^nC_r x^{n-r} a^r.$

2. $(1+x)^n = {}^nC_0 + {}^nC_1 \times x + {}^nC_2 \times x^2 + \ldots + {}^nC_r \times x^r + \ldots + {}^nC_n x^n$

$$\Rightarrow \quad (1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r.$$

In this expansion, we have

(i) total number of terms = $(n+1)$.

(ii) general term, $T_{r+1} = {}^nC_r x^r.$

3. $(x-a)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 - \ldots + (-1)^r \cdot {}^nC_r x^{n-r} a^r + \ldots + (-1)^n \cdot {}^nC_n x^n$

$$\Rightarrow \quad (x-a)^n = \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r x^{n-r} a^r.$$

In this expansion, we have

(i) total number of terms = $(n+1)$.

(ii) sum of the powers of *x* and *a* in each term is *n*.

(iii) general term is given by $T_{r+1} = (-1)^r \cdot {}^nC_r x^{n-r} a^r.$

4. $(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots + (-1)^r \cdot {}^nC_r x^r + \ldots + (-1)^n \cdot {}^nC_n x^n$

$\Rightarrow \quad (1-x)^n = \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r x^r.$

In this expansion, we have

(i) total number of terms $= (n+1)$.

(ii) general term, $T_{r+1} = (-1)^r \cdot {}^nC_r x^r$.

5. (i) $(x+a)^n + (x-a)^n = 2[{}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + {}^nC_4 x^{n-r} a^4 + \ldots]$.

(ii) $(x+a)^n - (x-a)^n = 2[{}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + {}^nC_5 x^{n-5} a^5 + \ldots]$.

6. Middle Terms

(i) If n is even then $\left(\frac{n}{2}+1\right)$th term is the middle term.

(ii) If n is odd then $\frac{1}{2}(n+1)$th and $\frac{1}{2}(n+3)$th terms are the middle terms.